INTERNATIONAL JOURNAL OF CREATIVE RESEARCH AND STUDIES
www.ijcrs.org ISSN-0249-4655

Further Remarks on the Numerical Range of λ-Commuting Operators

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Abstract
Let B(H) denote the Banach Algebra of all Bounded operators on a Complex Hilbert Space H. We consider operators A, B ∈ B(H) satisfying the property that AB =λBA , λ ∈ C, AB ≠ 0 and show that if A and B are self-ad joint then the numerical ranges of AB and BA are either real or pure imaginary and deduce related results.

Keywords: *Numerical range, λ-Commuting Operators, normal and self- ad joint operators*

INTRODUCTION

Let H be an infinite dimensional Hilbert space and B(H) denote the Banach Algebra of bounded linear operators on H ,then we say that A and B are λ-Commuting Operators if $AB = \lambda BA$, where λ is a complex number and $AB \neq 0$.This type of operators have been studied by a number of Authors including [1], [2], [3] and [5].

Commutating operators have had extensive applications in quantum mechanical observation and analysis of the spectra. We show that when two operators A and B commute to a scale factor then the numerical range $W(AB)$ of the product is real or pure imaginary. J.A Brook, P. Busch and D.B Pearson proved the following result using the spectral analysis approach.

Theorem A

Let $A, B \in B(H)$ be such that $AB = \lambda AB$, $AB \neq 0$ for a complex number λ. Then we have;

(i) If A or B is self adjoint then λ is a real number.

(ii) If both A and B are self adjoint then $\lambda \in \{1, -1\}$.

(iii) If both A and B are self adjoint and one of them is positive then $\lambda = 1$.

They also showed that if A and B λ commute then they have the same spectrum. In this paper apart from investigating the numerical range properties for AB and BA we use the Putnam- Fugelede theorem to provide an alternative proof of the theorem.

NOTATION AND TERMINOLOGY

Given an operator A on a Hilbert space H we denote the numerical range, spectrum and kernel of A by the symbols $W(A)$, $\sigma(A)$ and $ker(A)$ respectively.
Thus we have that:

$W(A) = \{(Ax, x); \|x\| = 1\}$
$\sigma(A) = \{\lambda; A - \lambda I \text{ is not invertible }\}$
$ker(A) = \{x \in H; Ax = 0\}$

We say that λ is a normal eigenvalue of A if $\{x \in H; Ax = \lambda x\} = \{x \in H; A*x = \bar{\lambda} x\}$ or equivalently if ker $(A - \lambda I) = ker(A* - \bar{\lambda} I)$

For any two operators A and B the commutator of A and B is denoted by $[A, B] = AB - BA$. An operator A is said to be normal if $AA^* = A^*A$, self-ad joint if $A = A^*$, and anti-self-ad joint if
$A = -A^*$.

We have the following inclusion; $\{Self\text{-}ad\ joint\} \subset \{Normal\}$ and $\{Anti - self\text{-}ad\ joint\} \subset \{Normal\}$

RESULTS

We first use the well-known Putnam Fueglede property to give an alternative proof of theorem A.

Lemma 1.1

Let A, B be bounded operators on a Hilbert space H such that $AB = \lambda BA$ and $AB \neq 0$ where λ is a complex number. Then:

If A or B is Self-ad joint we must have that λ is a real number
If A and B are Self ad joint then $\lambda \in \{1, -1\}$
If A and B are self-ad joint and either A or B is positive then $\lambda = 1$.

Proof

(i) $AB = \lambda BA$ if and only if
$B^*A^* = \bar{\lambda} A^*B^*$.
But if say $A = A^*$ then
$B*A = \bar{\lambda} AB^*$.
By Putnam Fueglede theorem $B^*A^* = \lambda A^*B^*$. Hence we have that
$\bar{\lambda} A^*B^* = \lambda A^*B^*$

Consequently we have that $(\bar{\lambda} - \lambda)A^*B^* = 0$ Since $AB \neq 0$ we obtain the result $(\bar{\lambda} - \lambda) = 0$. So
$\bar{\lambda} = \lambda$

(ii) If $A = A^*$, and $B = B^*$ then $AB = \lambda BA$ and so $(AB)^* = \bar{\lambda}\lambda BA = |\lambda|^2 BA$.

Hence,
$BA=\lambda\bar{\lambda}AB=\lambda\bar{\lambda}BA=|\lambda|^2BA$.
Consequently $(1-|\lambda|^2)BA=0$. So from (i) $\lambda=\pm1$.

(iii) Let $AB=-BA$ then Consider the commutator of A and B namely $AB-BA=AB+AB\neq0$.
However this leads to the anti-commutator being non zero. i.e. $AB+BA\neq 0$.
This is a contradiction hence we must have that $\lambda=1$.

Lemma 2.1
If an operator T is anti-self adjoint then W(T)is a pure imaginary.

Proof
Note that $\lambda\in W(T)\Leftrightarrow\lambda=(Tx,x)=(x,T^*x)=-(x,Tx)=-\bar{\lambda}$. Thus $\lambda+\bar{\lambda}=0$ and hence λ is a pure imaginary number.

Theorem 2.2
Let $AB=\lambda BA$, $AB\neq0$ and B be a normal operator. If A is anti-self-ad joint then λ is a real number.

Proof
If $AB=\lambda BA$ then we take the ad joint on both sides to find $B^*A^*=\bar{\lambda}A^*B^*$. Hence applying the Putnam-Fueglede theorem we have that

$B^*A=\lambda AB^*\Leftrightarrow-B^*A^*=-\lambda A^*B^*\Leftrightarrow(AB)^*=\lambda A^*B^*\Leftrightarrow(\lambda BA)^*=\lambda A^*B^*\Leftrightarrow\lambda A^*B^*\Leftrightarrow(\bar{\lambda}-\lambda)A^*B^*=0$. Hence $\bar{\lambda}=\lambda$

Corollary 2.3
If A and B are anti-self-ad joint operator such that $AB=\lambda BA$ and $AB\neq0$ then $\lambda\in\{1,-1\}$

Proof
Since A is normal we have that λ is a real number. Furthermore from AB = λBA we obtain
$B^*A^*=\lambda A^*B^*\Leftrightarrow BA=\lambda AB=\lambda^2BA\Leftrightarrow(1-(\lambda)^2)BA=0$.Since $AB\neq0$ we obtain $\lambda=\pm1$

Theorem 2.4
Let $A,B\in B(H)$ be such that $AB=\lambda BA$, $\lambda\in C$, $AB\neq 0$ where A and B are self-ad joint. Then
$W(AB)$ and $W(BA)$ are either real or pure imaginary.

Proof
Let $\mu\in W(AB)$.Then $\mu=(ABx,x)=(Bx,Ax)$. But since $W(AB)=W(\lambda BA)=\lambda W(BA)$ we must have that $\lambda(BAx,x)=\lambda(Ax,Bx)=\bar{\lambda}(Bx,Ax)=\lambda\bar{\mu}=\mu$
Now for A and B self-ad joint $\lambda=\pm1$ so that $\mu=\pm\bar{\mu}$

Corollary 2.5
Let $A,B\in B(H)$ be such that $AB=\lambda BA$, $\lambda\in C$, $AB\neq 0$,where A and B are self-ad joint. If μ is an eigen - value of AB then μ is either real or pure imaginary.

Proof

If μ is an eigenvalue of AB then $\mu \in W(AB)$. By the theorem above $\mu = \lambda\bar{\mu}$ But since A and B are self-ad joint $\lambda = \pm1$. So that $\mu = \pm\bar{\mu}$. Hence μ is either real or pure imaginary.

Corollary 2.6

Let $A, B \in B(H)$ be such that $AB = \lambda BA, \lambda \in C, AB \neq 0$ where A and B are Self-ad joint. Then every eigenvalue μ of AB or BA is normal.

Proof

Since A and B are self-ad joint we have that either $\lambda = \pm1$. Hence AB is either self-ad joint or anti self-ad joint. In both cases AB is normal. Hence μ is a normal eigenvalue.

Corollary 2.7

The eigen-spaces of the eigenvalues of AB or BA reduce AB or BA respectively where AB are seladjoint λ -commuting operators.

Theorem 2.8

Let $A, B \in B(H)$ be such that $AB = \lambda BA, AB \neq 0$ where A and B are anti-self –ad joint operators.
Then $W(AB)$ and $W(BA)$ are either pure imaginary or real.

Proof

Let $\mu \in W(AB)$.Then we have that

$$\mu = (ABx, x) = (Bx, A^*x) = -(Bx, Ax).$$

Also we have that

$$\mu = \lambda(BAx, x) = -\lambda(Ax, Bx) = -\bar{\lambda}(Bx, Ax)$$

Therefore $-(Bx, Ax) = -\bar{\lambda}(Bx, Ax)$ i.e. $\mu = \lambda\bar{\mu}$. When A and B are anti-self-ad joint by corollary 1 we must have $\lambda = \pm1$. Thus $\mu = \pm\bar{\mu}$

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